THE APPLICATION OF THE MOVING FRAME METHOD TO INTEGRAL GEOMETRY IN THE HEISENBERG GROUP

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Abstract. We show the fundamental theorems of curves and surfaces in the 3-dimensional Heisenberg group and find a complete set of invariants for curves and surfaces respectively. The proof is based on the Cartan's method of moving frames and Lie group theory. As an application of the main theorem, a Croton-type formula is proved in terms of p-area which naturally arises from the variation of volume. The application makes a connection between CR Geometry and Integral Geometry.

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1. Introduction

In Euclidean spaces, the fundamental theorem of curves states that any unit-speed curve is completely determined by its curvature and torsion. More precisely, given two functions $k(s)$ and $\tau(s)$ with $k(s) > 0$, there exists a unit-speed curve whose curvature and torsion are the functions k and τ, respectively, uniquely up to a Euclidean rigid motion. We present the analogous theorems of curves and surfaces in the 3-dimensional Heisenberg group H_1. The main task in the paper also includes the understanding to the structure of the group of transformations in H_1, which is similar to the group of rigid motions in Euclidean spaces. Moreover, we also develop the concept of the invariants for curves and surfaces in above sense. It should be emphasised that owning such invariants helps us understand the geometric structure in CR manifolds and develop the application to the field of Integral Geometry.

We give a brief review of the Heisenberg group. All the details can be found in [1]. The Heisenberg group H_1 is the space $\mathbb{R}^3$ associated with the group multiplication

$$(x_1, y_1, z_1) \circ (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + y_1x_2 - x_1y_2), \tag{1.1}$$

which is also a 3-dimensional Lie group. The standard left-invariant vector fields in H_1

$$\mathring{e}_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \mathring{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad \text{and} \quad T = \frac{\partial}{\partial z} \tag{1.2}$$

form a basis of the vector space of left-invariant vector fields, where $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ denotes the standard basis in $\mathbb{R}^3$. The standard contact bundle $\xi := span\{\mathring{e}_1, \mathring{e}_2\}$ in H_1 is a subbundle of the tangent bundle TH_1. Equivalently, the contact bundle can be defined as

$$\xi = \ker \Theta,$$

where

$$\Theta = dz + xdy - ydx \tag{1.3}$$

is the standard contact form. The CR structure on H_1 is the endomorphism $J : \xi \to \xi$ defined by

$$J(\mathring{e}_1) = \mathring{e}_2 \quad \text{and} \quad J(\mathring{e}_2) = -\mathring{e}_1. \tag{1.4}$$

The Heisenberg group H_1 can be regarded as a pseudo-hermitian manifold by considering H_1 associated with the standard pseudo-hermitian structure (J, Θ). Recall that a pseudo-hermitian transformation on H_1 is a diffeomorphism on H_1 preserving the pseudo-hermitian structure (J, Θ). For more information about pseudo-hermitian structure, we refer the readers to [3][12][13][18]. Denote $PSH(1)$ be the group

of pseudo-hermitian transformations on H_1, and call the element in $PSH(1)$ a **symmetry**. A symmetry in H_1 plays the same role as the rigid motion in $\mathbb{R}^n$ and will be characterized in Subsection 3.1.

Let $\gamma : I \to H_1$ be a parametrized curve. For each $t \in I$, the velocity $\gamma^{'}(t)$ has the natural decomposition

$$\gamma^{'}(t) = \gamma^{'}_{\xi}(t) + \gamma^{'}_{T}(t), \tag{1.5}$$

where $\gamma^{'}_{\xi}(t)$ and $\gamma^{'}_{T}(t)$ are, respectively, the orthogonal projection of $\gamma^{'}(t)$ on ξ along T and the orthogonal projection of $\gamma^{'}(t)$ on T along ξ.

Definition 1.1. A **horizontally regular curve** is a parametrized curve $\gamma(t)$ such that $\gamma^{'}_{\xi}(t) \neq 0$ for all $t \in I$. We say $\gamma(t)$ is a **horizontal curve** if $\gamma^{'}(t) = \gamma^{'}_{\xi}(t)$ for all $t \in I$.

From the approach of Contact Geometry, some authors call the horizontally regular cuvers the *Legendrian curves*, for examples, in [10, 14, 8]. Proposition 4.1 shows that a horizontally regular curve can always be reparametrized by the parameter s satisfying $|\gamma^{'}_{\xi}(s)| = 1$ for all s. Such a curve is called **with horizontal unit-speed** and the parameter s is called **the horizontal arc-length** for $\gamma(s)$, which is unique up to a constant. Through the article the length of the vectors $|\cdot|$ and the inner product $\langle \cdot, \cdot \rangle$ are defined with respect to the Levi-metric, and the orthonormality of the vectors $\mathring{e}_1$ and $\mathring{e}_2$ on the contact plane ξ is always in the sense of the Levi-metric.

For a horizontally regular curve $\gamma(s)$ parameterized by the horizontal arc-length s, we define the **p-curvature** $k(s)$ and the **contact normality** $\tau(s)$ by

$$\begin{aligned} k(s) &:= \langle \frac{dX(s)}{ds}, Y(s) \rangle, \\ \tau(s) &:= \langle \gamma^{'}(s), T \rangle, \end{aligned} \tag{1.6}$$

where $X(s) = \gamma^{'}_{\xi}(s)$ and $Y(s) = JX(s)$. Notice that $k(s)$ is analogous to the curvature of the curve in $\mathbb{R}^n$, while $\tau(s)$ measures how far the curve is from being horizontal. We also point out that $k(s)$ and $\tau(s)$ are invariant under pseudo-hermitian transformations of horizontally regular curves.

The first theorem of the paper says that horizontally regular curves are completely prescribed by the functions $k(s)$ and $\tau(s)$.

Theorem 1.2 (The fundamental theorem for curves in H_1). *Given C^1-functions $k(s), \tau(s)$, there exists a horizontally regular curve $\gamma(s)$ with horizontal unit-speed having $k(s)$ and $\tau(s)$ as its p-curvature and contact normality, respectively. In addition, any regular curve $\widetilde{\gamma}(s)$ with*

horizontal unit-speed satisfying the same p-curvature $k(s)$ and contact normality $\tau(s)$ differs from $\gamma(s)$ by a pseudo-hermitian transformation $g \in PSH(1)$, namely,

$$\widetilde{\gamma}(s) = g \circ \gamma(s) \tag{1.7}$$

for all s.

Since a curve $\gamma(s)$ is horizontal if and only if the contact normality $\tau(s) = 0$, we immediately have the following corollary.

Corollary 1.3. *Given a C^1-function $k(s)$, there exists a horizontal curve $\gamma(s)$ with horizontal unit-speed having $k(s)$ as its p-curvature. In addition, any horizontal curve $\widetilde{\gamma}(s)$ with horizontal unit-speed satisfying the same p-curvature differs from $\gamma(s)$ by a pseudo-hermitian transformation $g \in PSH(1)$, namely,*

$$\widetilde{\gamma}(s) = g \circ \gamma(s)$$

for all s.

If the horizontally regular curve γ is not parameterized by horizontal arc-length, we show, in Subsection 4.2, the explicit formulae for the p-curvature and the contact normality.

Theorem 1.4. *Let $\gamma(t) = \big(x(t), y(t), z(t)\big) \in H_1$ be a horizontally regular curve, not necessarily with horizontal unit-speed. The p-curvature $k(t)$ and the contact normality $\tau(t)$ of $\gamma(s)$ are*

$$\begin{aligned} k(t) &= \frac{x^{'}y^{''} - x^{''}y^{'}}{\big((x^{'})^2 + (y^{'})^2\big)^{\frac{3}{2}}}(t), \\ \tau(t) &= \frac{xy^{'} - x^{'}y + z^{'}}{\big((x^{'})^2 + (y^{'})^2\big)^{\frac{1}{2}}}(t). \end{aligned} \tag{1.8}$$

Notice that in (1.8) the p-curvature $k(t)$ depends only on $x(t)$, $y(t)$. We observe that $k(t)$ is the signed curvature of the plane curve $\alpha(t) := \pi \circ \gamma(t) = \big(x(t), y(t)\big)$, where π is the projection onto the xy-plane along the z-axis. It is the fact that the signed curvature of a given plane curve completely describes the curve's behavior, we have the corollary:

Corollary 1.5. *Suppose two horizontally regular curves in H_1 differ by a Heisenberg rigid motion, then their projections onto the xy-plane along the z-axis differ by a Euclidean rigid motion. In particular, two horizontal curves in H_1 differ by a Heisenberg rigid motion if and only if they are congruent in the Euclidean plane.*

As an example, we calculate the p-curvature and contact normality for the geodesics, and obtain the characteristic description of the geodesics.

THEOREM 1.6. *In H_1, the geodesics are the horizontally regular curves with constant p-curvature and zero contact normality.*

In the second part of the paper, the fundamental theorem of surfaces in H_1 will be established. For an embedded regular surface $\Sigma \subset H_1$, recall that a singular point $p \in \Sigma$ is a point such that the tangent plane $T_p\Sigma$ coincides with the contact plane ξ_p at p. Therefore outside the singular set (the non-singular part of Σ), the line bundle $T\Sigma \cap \xi$ forms a one-dimensional foliation, which is called the *characteristic foliation*.

DEFINITION 1.7. Let $F : U \to H_1$ be a parameterized surface with coordinates (u, v) on $U \subset \mathbb{R}^2$. We say F is a **normal parametrization** if

(1) $F(U)$ is a surface without singular points,
(2) $F_u := \frac{\partial F}{\partial u}$ defines the characteristic foliation on $F(U)$,
(3) $|F_u| = 1$ for each point $(u, v) \in U$, where the norm is with respect to the Levi-metric.

We call (u, v) **normal coordinates** of the surface $F(U)$.

It is easy to see that normal coordinates always exist locally near a non-singular point $p \in \Sigma$. In addition, for a normal parameterized surface F, denote $X = F_u$, $Y = JX$ and $T = \frac{\partial}{\partial z}$. We define smooth functions a, b, c, l and m on U by

$$\begin{aligned} &a := \langle F_v, X\rangle, \quad b := \langle F_v, Y\rangle, \qquad c =: \langle F_v, T\rangle, \\ &l := \langle F_{uu}, Y\rangle, \quad m := \langle F_{uv}, Y\rangle, \end{aligned} \tag{1.9}$$

and call a, b and c *the coefficients of the first kind* of F, and l, m *the coefficients of the second kind.* All coefficients satisfy the integrability conditions

$$\begin{aligned} &a_u = bl, \quad b_u = -al + m, \quad c_u = 2b, \\ &l_v - m_u = 0, \end{aligned} \tag{1.10}$$

where the subscripts denote the partial derivatives.

The following theorem states that these coefficients are the complete differential invariants for the map F.

THEOREM 1.8. *Let $U \subset \mathbb{R}^2$ be a simply connected open set. Suppose that a, b, c, l and m are functions on U satisfying the integrability condition (1.10). Then there exists a normal parameterized surface $F : U \to H_1$ having a, b, c and l, m as the coefficients of first kind*

and second kind of F, respectively. In addition, any $\widetilde{F} : U \to H_1$ normal parameterized surface with the same coefficients of first kind and second kind differ from F by a Heisenberg rigid motion, namely, $\widetilde{F}(u,v) = g \circ F(u,v)$ for all $(u,v) \in U$ for some $g \in PSH(1)$.

In (5.24), we will show that the function l, up to a sign, is independent of the choice of normal coordinates, and hence it is a differential invariant of the surface $F(U)$. Actually l is the *p-mean curvature*[4]. In particular, $F(U)$ is a p-minimal surface when $l = 0$. Such a parametrization $F : U \to H_1$ is called a *normal p-minimal parameterized surface.* In this case, the integrability condition (1.10) becomes

$$a_u = 0, \quad b_{uu} = 0, \quad c_u = 2b, \tag{1.11}$$

$$m = b_u, \tag{1.12}$$

and the coefficients of first kind completely dominate those of second kind. We conclude all above as the following result.

THEOREM 1.9. *Let $U \subset \mathbb{R}^2$ be a simply connected open set. Suppose that a, b and c are smooth functions on U satisfying the integrability condition (1.11). Then there exists a normal p-minimal parameterized surface $F : U \to H_1$ having a, b and c as the coefficients of first kind of F, which also determines the coefficient b of the second kind as in (1.12). In addition, any normal p-minimal parameterized surface $\widetilde{F} : U \to H_1$ with the same conditions differs from F by a Heisenberg rigid motion, namely, $\widetilde{F}(u,v) = g \circ F(u,v)$ in U for some $g \in PSH(1)$.*

In Section 5, we will obtain the other invariants on $F(U)$: $\alpha := \frac{b}{c}$ (up to a sign, called the *p-variation*), and the restricted adapted metric $g_\Theta|_\Sigma$ on the surface Σ. Actually α is the function such that the vector field $\alpha e_2 + T$ is tangent to the surface, where $e_2 = Je_1$ and e_1 is a unit vector field tangent to the characteristic foliation. Let e_Σ be the unit vector field tangent to the surface

$$e_\Sigma = \frac{\alpha e_2 + T}{\sqrt{1+\alpha^2}},$$

then we observe that these invariants α, l, e_Σ satisfy the integrability condition:

$$\begin{aligned}(1+\alpha^2)^{\frac{3}{2}}(e_\Sigma l) = {} & (1+\alpha^2)(e_1 e_1 \alpha) - \alpha(e_1\alpha)^2 + 4\alpha(1+\alpha^2)(e_1\alpha) \\ & + \alpha(1+\alpha^2)^2 K + \alpha l (1+\alpha^2)^{\frac{1}{2}}(e_\Sigma \alpha) + \alpha(1+\alpha^2) l^2,\end{aligned} \tag{1.13}$$

where K is the Gaussian curvature with respect to $g_\Theta|_\Sigma$.

After studying the invariants in H_1, we show the second main theorem which says that the three invariants: the Riemannian metric (induced by the adapted metric), the p-mean curvature, and the p-variation comprise a complete set of invariants for a surface without singular points.

THEOREM 1.10 (The fundamental theorem for surfaces in H_1). *Let (Σ, g) be a 2-dimensional Riemannian manifold with Guassian curvature K, and let α', l' be two real-valued functions on Σ. Assume that K, $\alpha^{'}$ and $l^{'}$ satisfy the integrability condition (1.13). Then for every non-singular point $p \in \Sigma$, there exists an open neighborhood U containing p and an embedding $f : U \to H_1$ such that*

$$\begin{aligned} g &= f^*(g_\Theta), \\ \alpha^{'} &= f^*\alpha, \\ l' &= f^*l, \end{aligned}$$

where α, l are the induced p-variation and p-mean curvature on $f(U)$. Moreover, f is unique up to a Heisenberg rigid motion.

The third part of the paper is an application of the motion equations and the structure equations we obtain for the proof of fundamental theorems. We will derive the Crofton formmula in H_1 which is a classic result of Integral Geometry [15][16], relating the length of a fixed curve and the number of intersections for the curve and randomly oriented lines passing through it. In $\mathbb{R}^2$, given a fixed piecewise regular curve γ, the Crofton formula states that

$$\int_{l\cap\gamma\neq\emptyset} n(l\cap\gamma)dL = 4\cdot length(\gamma),$$

where dL is the kinematic density defined on the set of oriented lines in $\mathbb{R}^2$, and $n(l\cap\gamma)$ is the number of intersections of the line l with γ. We have the analogues formula in H_1. Of particular interest is that the geometric quantity on one side is the p-area which naturally arises from the variation of the volume for domains in CR manifolds [4].

THEOREM 1.11 (Croton formula in H_1). *Suppose $\mathbb{X} : (u, v) \in \Omega \mapsto \Sigma \subset H_1$ is a regular surface for some domain $\Omega \subset \mathbb{R}^2$. Let $\mathcal{L}$ be the set of oriented horizontal lines in H_1 and $n(l \cap \Sigma)$ be the number of intersections of the horizontal line $l \in \mathcal{L}$ with the surface Σ. Then we have the Crofton formula in H_1*

$$\int_{l\in\mathcal{L},\ l\cap\Sigma\neq\emptyset} n(l\cap\Sigma)dL = 4\cdot \text{p-area}(\Sigma),$$

where $dL := dp \wedge d\theta \wedge dt$ is the kinematic density on $\mathcal{L}$.

We give the outline of the paper. In section 2, we state two propositions about the existence and uniqueness of mappings from a smooth manifold into a Lie group G, which underlies our main theorems. In section 3, we not only express the representation of $PSH(1)$ but discuss how the matrix Lie group $PSH(1)$ can be interpreted as the set of moving frames on the homogeneous space $H_1 = PSH(1)/SO(2)$; the moving frame formula in H_1 via the (left-invariant) Maurer-Cartan form will also be derived. In section 4, we compute the Darboux derivative of the lift of a horizontally regular curve and give the proof of the first main theorem; moreover, we calculate the p-curvature and the contact normality of horizontally regular curves and geodesics. In section 5, we compute the Darboux derivative of the lift of normal parameterized surfaces. By deriving the formula of change of coordinates, we achieve the complete set of differential invariants for a normal parameterized surface. In section 6, by calculating the Darboux derivative of the lift of $f : \Sigma \to H_1$, we show the fundamental theorem for surfaces Σ in H_1. In section 7, one of the applications for the fundamental theorem of curves, the Crofton formula, will be shown, which connects CR geometry and Integral Geometry.

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2. Calculus on Lie groups

We recall two basic theorems from Lie groups, which play the essential roles in the proof of the main theorems. For the details we refer the readers to [2][7][10][11][17].

Given a connected smooth manifold M. Let $G \subset GL(n, R)$ be the matrix Lie group with Lie algebra $\mathfrak{g}$ and the (left-invariant) Maurer-Cartan form ω. We first introduce the theorem of uniqueness.

Theorem 2.1. *Given two maps $f, \widetilde{f} : M \to G$, then $\widetilde{f}^*\omega = f^*\omega$ if and only if $\widetilde{f} = g \cdot f$ for some $g \in G$.*

We call the Lie algebra-valued 1-form $f^*\omega$ *the Darboux derivative* of the map $f : M \to G$.

The second result is the theorem of existence.

THEOREM 2.2. *Suppose that ϕ is a $\mathfrak{g}$-valued 1-form on a simply connected manifold M. Then there exists a map $f : M \to G$ satisfying $f^*\omega = \phi$ if and only if $d\phi = -\phi \wedge \phi$. Moreover, the resulting map f is unique up to a group action.*

We mention that the proof of Theorem 2.2 relies on the Frobenius theorem.

3. The group of pseudo-hermitian transformations on H_1

3.1. **The pseudo-hermitian transformations on H_1.** A pseudo-hermitian transformation on H_1 is a diffeomorphism Φ on H_1 preserving the CR structure J and the contact form Θ; it satisfies

$$\Phi_* J = J \Phi_* \text{ on } \xi \quad \text{and} \quad \Phi^*\Theta = \Theta \text{ in } H_1. \tag{3.1}$$

The trivial example of the pseudo-hermitian transformation is the left translation L_p in H_1; the other example is defined by $\Phi_R : H_1 \to H_1$

$$\Phi_R \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{3.2}$$

where $R \in SO(2)$ is a $2{\times}2$ special orthogonal matrix.

Let $PSH(1)$ be the group of pseudo-hermitian transformations on H_1. We shall show that the group $PSH(1)$ exactly consists of all the transformations of the forms $\Phi_{p,R} := L_p \circ \Phi_R$, a transformation Φ_R followed by a left translation L_p. More precisely, we have

$$\Phi_{p,R} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + by + p_1 \\ cx + dy + p_2 \\ (ap_2 - cp_1)x + (bp_2 - dp_1)y + z + p_3 \end{pmatrix}, \tag{3.3}$$

where $p = (p_1, p_2, p_3)^t \in H_1$ and $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2)$.

THEOREM 3.1. *Let $\Phi : H_1 \to H_1$ be a pseudo-hermitian transformation. Then $\Phi = L_p \circ \Phi_R$ for some $R \in SO(2)$ and $p \in H_1$.*

Proof. It suffices to consider the pseudo-hermitian transformation $\Phi : H_1 \to H_1$ such that $\Phi(0) = 0$. Indeed, if $\Phi(0) = p$ for some $p \in H_1 \backslash \{0\}$, then the composition $L_{p^{-1}} \circ \Phi$ is a transformation fixing the origion. Therefore, we reduce the proof of Theorem 3.1 to the following Lemma:

LEMMA 3.2. *Let Φ be a pseudo-hermitian transformation on H_1 such that $\Phi(0) = 0$. Then, for any $p \in H_1$, the matrix representation of*

$\Phi_*(p)$ with respect to the standard basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ of $\mathbb{R}^3$ is

$$\Phi_*(p) = \begin{pmatrix} \cos\alpha_0 & -\sin\alpha_0 & 0 \\ \sin\alpha_0 & \cos\alpha_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})}, \tag{3.4}$$

for some real constant α_0 which is independent of p. Thus Φ_* is a constant matrix.

To prove Lemma 3.2, we calculate the matrix representation of $\Phi_*(p)$ with respect to the basis $(\mathring{e}_1, \mathring{e}_2, T)$. For $i = 1, 2$,

$$\Theta\left(\Phi_*\mathring{e}_i\right) = \left(\Phi^*\Theta\right)\left(\mathring{e}_i\right) = \Theta\left(\mathring{e}_i\right) = 0,$$

we see that the contact bundle ξ is invariant under Φ_*. In addition, let h be the Levi metric on ξ defined by $h(X, Y) = d\Theta(X, JY)$, then

$$\begin{aligned} \Phi^*h(X, Y) &= h(\Phi_*X, \Phi_*Y) = d\Theta(\Phi_*X, J\Phi_*Y) \\ &= d\Theta(\Phi_*X, \Phi_*JY) = \Phi^*(d\Theta)(X, JY) \\ &= d(\Phi^*\Theta)(X, JY) = d\Theta(X, JY) = h(X, Y). \end{aligned}$$

Hence $h\left(\Phi_*X, \Phi_*Y\right) = h\left(X, Y\right)$ for every $X, Y \in \xi$. Thus Φ_* is orthogonal on ξ. On the other hand,

$$\Theta(\Phi_*T) = \Theta\left(\Phi_*\frac{\partial}{\partial z}\right) = \left(\Phi^*\Theta\right)\left(\frac{\partial}{\partial z}\right) = \Theta\left(\frac{\partial}{\partial z}\right) = 1,$$

and, for all $X \in \xi$,

$$\begin{aligned} d\Theta(X, \Phi_*T) &= d\Theta(\Phi_*\Phi_*^{-1}X, \Phi_*T) = (\Phi^*d\Theta)(\Phi_*^{-1}X, T) \\ &= (d\Phi^*\Theta)(\Phi_*^{-1}X, T) = d\Theta(\Phi_*^{-1}X, T) = 0. \end{aligned}$$

By the uniqueness of the characteristic vector fields, we have $\Phi_*T = T$. From the above argument, we conclude that

$$\Phi_*(p) = \begin{pmatrix} \cos\alpha\left(p\right) & -\sin\alpha\left(p\right) & 0 \\ \sin\alpha\left(p\right) & \cos\alpha\left(p\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\left(\mathring{e}_1, \mathring{e}_2, \frac{\partial}{\partial z}\right)},$$

for some real-valued function α on H_1.

Next, we rewrite the matrix representation of $\Phi_*(p)$ from the basis $\left(\mathring{e}_1, \mathring{e}_2, \frac{\partial}{\partial z}\right)$ to the basis $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Let $\Phi = (\Phi^1, \Phi^2, \Phi^3)$, $p = (p_1, p_2, p_3)$, $\mathring{e}_1\left(p\right) = \frac{\partial}{\partial x} + p_2\frac{\partial}{\partial z}$ and $\mathring{e}_2\left(p\right) = \frac{\partial}{\partial y} - p_1\frac{\partial}{\partial z}$, then

$$\begin{aligned}\Phi_*(p)\left(\frac{\partial}{\partial x}\right) &= \Phi_*(p)\left[\mathring{e}_1\,(p) - p_2\frac{\partial}{\partial z}\right] = \Phi_*(p)\left[\mathring{e}_1\,(p)\right] - p_2\frac{\partial}{\partial z}\\ &= \cos\alpha\,(p)\,\mathring{e}_1\left[\Phi(p)\right] + \sin\alpha\,(p)\,\mathring{e}_2\left[\Phi(p)\right] - p_2\frac{\partial}{\partial z}\\ &= \cos\alpha\,(p)\,\frac{\partial}{\partial x} + \sin\alpha\,(p)\,\frac{\partial}{\partial y}\\ &\quad + \left[\cos\alpha\,(p)\,\Phi^2\,(p) - \sin\alpha\,(p)\,\Phi^1\,(p) - p_2\right]\frac{\partial}{\partial z},\end{aligned}$$

and

$$\begin{aligned}\Phi_*(p)\left(\frac{\partial}{\partial y}\right) &= \Phi_*(p)\left[\mathring{e}_2\,(p) + p_1\frac{\partial}{\partial z}\right] = \Phi_*(p)\left[\mathring{e}_2\,(p)\right] + p_1\frac{\partial}{\partial z}\\ &= -\sin\alpha\,(p)\,\mathring{e}_1\left[\Phi(p)\right] + \cos\alpha\,(p)\,\mathring{e}_2\left[\Phi(p)\right] + p_1\frac{\partial}{\partial z}\\ &= -\sin\alpha\,(p)\,\frac{\partial}{\partial x} + \cos\alpha\,(p)\,\frac{\partial}{\partial y}\\ &\quad + \left[-\sin\alpha\,(p)\,\Phi^2\,(p) - \cos\alpha\,(p)\,\Phi^1\,(p) + p_1\right]\frac{\partial}{\partial z}.\end{aligned}$$

Thus,

(3.5)

$$\Phi_*(p) = \begin{pmatrix}\cos\alpha\,(p) & -\sin\alpha\,(p) & 0\\ \sin\alpha\,(p) & \cos\alpha\,(p) & 0\\ \Phi^3_x(p) & \Phi^3_y(p) & 1\end{pmatrix}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right),} := \begin{pmatrix}\Phi^1_x & \Phi^1_y & \Phi^1_z\\ \Phi^2_x & \Phi^2_y & \Phi^2_z\\ \Phi^3_x & \Phi^3_y & \Phi^3_z\end{pmatrix},$$

where

$$\begin{aligned}\Phi^3_x(p) &:= \frac{\partial\Phi^3_x}{\partial x} = \cos\alpha\,(p)\,\Phi^2\,(p) - \sin\alpha\,(p)\,\Phi^1\,(p) - p_2,\\ \Phi^3_y(p) &:= \frac{\partial\Phi^3_y}{\partial y} = -\sin\alpha\,(p)\,\Phi^2\,(p) - \cos\alpha\,(p)\,\Phi^1\,(p) + p_1,\end{aligned} \tag{3.6}$$

and denote the subscripts as the partial derivatives for all Φ^i's. By (3.5) that $\Phi^1_z = \Phi^2_z = 0$, it follows that the functions Φ^1 and Φ^2 both depend only on x and y, and so is α. Moreover, use (3.5) again and the facts $\Phi^1_{xy} = \Phi^1_{yx}$ and $\Phi^2_{xy} = \Phi^2_{yx}$, we have

$$\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\alpha_x\\ \alpha_y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix},$$

which implies that $\alpha_x = \alpha_y = 0$. Thus α is a constant on H_1, say $\alpha = \alpha_0$. From (3.5) and notice that $\Phi(0) = 0$, we finally get

$$\begin{aligned}\Phi^1 &= x\cos\alpha_0 - y\sin\alpha_0,\\ \Phi^2 &= x\sin\alpha_0 + y\cos\alpha_0,\end{aligned}$$

which implies that $\Phi^3_x = \Phi^3_y = 0$. Therefore

$$\Phi_*(p) = \begin{pmatrix}\cos\alpha_0 & -\sin\alpha_0 & 0\\ \sin\alpha_0 & \cos\alpha_0 & 0\\ 0 & 0 & 1\end{pmatrix}_{\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)}.$$

and the result follows. $\square$

3.2. **Representation of $PSH(1)$.** The pseudo-hermitian transformation $\Phi_{p,R}$ and the points $(x, y, z)^t$ in H_1 can be respectively represented as

$$\Phi_{p,R} \leftrightarrow M = \begin{pmatrix}1 & 0 & 0 & 0\\ p_1 & a & b & 0\\ p_2 & c & d & 0\\ p_3 & ap_2 - cp_1 & bp_2 - dp_1 & 1\end{pmatrix}, \tag{3.7}$$

and

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} \leftrightarrow X = \begin{pmatrix}1\\ x\\ y\\ z\end{pmatrix} \tag{3.8}$$

satisfying

$$MX = \begin{pmatrix}1\\ \Phi_{p,R}\begin{pmatrix}x\\ y\\ z\end{pmatrix}\end{pmatrix}. \tag{3.9}$$

Therefore, $PSH(1)$ can be represented as a matrix group

$$PSH(1) = \left\{ M \in GL(4,R) \;\middle|\; M = \begin{pmatrix}1 & 0 & 0 & 0\\ p_1 & a & b & 0\\ p_2 & c & d & 0\\ p_3 & ap_2 - cp_1 & bp_2 - dp_1 & 1\end{pmatrix}\right\}. \tag{3.10}$$

Let $psh(1)$ be the Lie algebra of $PSH(1)$. It is easy to see that the element of $psh(1)$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ x_1 & 0 & -{x_1}^2 & 0 \\ x_2 & {x_1}^2 & 0 & 0 \\ x_3 & x_2 & -x_1 & 0 \end{pmatrix}. \tag{3.11}$$

and the corresponding Maurer-Cartan form of $PSH(1)$ is of the form

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -{\omega_1}^2 & 0 \\ \omega^2 & {\omega_1}^2 & 0 & 0 \\ \omega^3 & \omega^2 & -\omega^1 & 0 \end{pmatrix}, \tag{3.12}$$

where ω_1^2 and $\omega^j, j = 1, 2, 3$ are 1-forms on $PSH(1)$.

3.3. **The oriented frames on H_1.** The oriented frame $(p; X, Y, T)$ on H_1 consists of the point $p \in H_1$ and the orthonormal vector fields $X \in \xi_p$, $Y = JX$ with respect to the Levi metric. We can identify $PSH(1)$ with the space of all oriented frames on H_1 as follows:

(3.13)

$$PSH(1) \ni M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p_1 & a & b & 0 \\ p_2 & c & d & 0 \\ p_3 & ap_2 - cp_1 & bp_2 - dp_1 & 1 \end{pmatrix} \leftrightarrow (p; X, Y, T),$$

where

$$\begin{aligned} p &= (p_1, p_2, p_3)^t, \\ X &= a\frac{\partial}{\partial x} + c\frac{\partial}{\partial y} + (ap_2 - cp_1)\frac{\partial}{\partial t}, \\ Y &= b\frac{\partial}{\partial x} + d\frac{\partial}{\partial y} + (bp_2 - dp_1)\frac{\partial}{\partial t}. \end{aligned} \tag{3.14}$$

Actually, we have that $X = a\mathring{e}_1(p) + c\mathring{e}_2(p)$ and $Y = b\mathring{e}_1(p) + d\mathring{e}_2(p)$, hence M is the unique 4×4 matrix such that

$$(p; X, Y, T) = (0; \mathring{e}_1, \mathring{e}_2, T)M. \tag{3.15}$$

3.4. **Moving frame formula.** Since $PSH(1)$ is a matrix Lie group, the Maurer-Cartan form has to be $\omega = M^{-1}dM$ or $dM = M\omega$ (see [5]). Immediately one has that

$$(dp; dX, dY, dT) = (p; X, Y, T) \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -{\omega_1}^2 & 0 \\ \omega^2 & {\omega_1}^2 & 0 & 0 \\ \omega^3 & \omega^2 & -\omega^1 & 0 \end{pmatrix}. \tag{3.16}$$

Thus, we have reached the moving frame formula:

$$
\begin{aligned}
dp &= \omega^1 X + \omega^2 Y + \omega^3 T, \\
dX &= \ \omega_1{}^2 Y + \omega^2 T, \\
dY &= -\omega_1{}^2 X - \omega^1 T, \\
dT &= 0.
\end{aligned} \tag{3.17}
$$

4. Differential invariants of horizontally regular curves in H_1

Proposition 4.1. *Any horizontally regular curve $\gamma(t)$ can be reparametrized by its horizontal arc-length s such that $|\gamma'_\xi(s)| = 1$.*

Proof. Define $s(t) = \int_0^t |\gamma'_\xi(u)| du$. Then any horizontal arc-length differs s up to a constant. By the fundamental theorem of calculus, we have $\frac{ds}{dt} = |\gamma'_\xi(t)|$. Since

$$
\frac{d\gamma}{ds} = \frac{d\gamma}{dt}\frac{dt}{ds} = \frac{\gamma'(t)}{|\gamma'_\xi(t)|}, \tag{4.1}
$$

we have $\gamma'_\xi(s) = \frac{\gamma'_\xi(t)}{|\gamma'_\xi(t)|}$, that is $|\gamma'_\xi(s)| = 1$. □

Definition 4.2. A **lift** of a mapping $f : M \to G/H$ is defined to be a map $F : M \to G$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & G \arrow[d] \\
M \arrow[ur, "F"] \arrow[r, "f"'] & G/H
\end{tikzcd}
$$

where G is a Lie group, H is a closed Lie subgroup and G/H is a homogeneous space. In additional, the other lift $\tilde{F}$ of f has to satisfy

$$
\tilde{F}(x) \quad = \quad F(x)g(x)
$$

for some map $g : M \to H$.

Remark 4.3. In the next section, we shall set $G = PSH(1)$, $M = (a, b) \subset R$, $f = \gamma$, $F = \tilde{\gamma}$, $G/H = PSH(1)/SO(2)$, and identify $PSH(1)/SO(2)$ with H_1.

4.1. **The Proof of Theorem 1.2.** By Proposition 4.1, we may assume that the horizontally regular curve $\gamma(s)$ is parametrized by the horizontal arc-length s. Each point on γ uniquely defines the oriented frame

$$(\gamma(s); X(s), Y(s), T), \tag{4.2}$$

where $X(s) = \gamma_\xi'(s)$ is the horizontally tangent vector of $\gamma(s)$ and $Y(s) = JX(s)$. By Remark 4.3, there exists the lift $\tilde{\gamma} \in PSH(1)$ of γ, which is unique up to a $SO(2)$ group action, and is still denoted by the same notation

$$\widetilde{\gamma}(s) = (\gamma(s); X(s), Y(s), T).$$

Let ω be the Maurer-Cartan form of $PSH(1)$. We shall derive the Darboux derivative $\widetilde{\gamma}^*\omega$ of the lift $\widetilde{\gamma}(s)$:

By the moving frame formula (3.17),

$$\begin{aligned} d\widetilde{\gamma}(s) &= \widetilde{\gamma}^* dp \\ &= X(s)\widetilde{\gamma}^*\omega^1 + Y(s)\widetilde{\gamma}^*\omega^2 + T\widetilde{\gamma}^*\omega^3. \end{aligned} \tag{4.3}$$

We also observe that all pull-back 1-forms by $\widetilde{\gamma}$ are the multiples of ds,

$$\begin{aligned} d\widetilde{\gamma}(s) &= \gamma_\xi'(s)ds + \gamma_T'(s)ds \\ &= X(s)ds + \gamma_T'(s)ds. \end{aligned} \tag{4.4}$$

Comparing (4.3) and (4.4), we have

$$\begin{aligned} &\widetilde{\gamma}^*\omega^1 = ds, \\ &\widetilde{\gamma}^*\omega^2 = 0, \\ &\widetilde{\gamma}^*\omega^3 = \langle \gamma'(s), T\rangle ds = \tau(s)ds. \end{aligned} \tag{4.5}$$

Insert $\widetilde{\gamma}^*\omega^3$ into (3.17),

$$dX(s) = Y(s)\widetilde{\gamma}^*{\omega_1}^2 + T\widetilde{\gamma}^*\omega^2 = Y(s)\widetilde{\gamma}^*{\omega_1}^2, \tag{4.6}$$

we get

$$\widetilde{\gamma}^*{\omega_1}^2 = \langle \frac{dX(s)}{ds}, Y(s)\rangle ds = k(s)ds. \tag{4.7}$$

As a consequence we obtain the Darboux derivative of $\widetilde{\gamma}$

$$\widetilde{\gamma}^*\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -k(s) & 0 \\ 0 & k(s) & 0 & 0 \\ \tau(s) & 0 & -1 & 0 \end{pmatrix} ds. \tag{4.8}$$

For any functions $k(s)$ and $\tau(s)$ defined on an open interval I. Suppose φ is the $psh(1)$-valued 1-form defined by (4.8). It is easy to check

that φ satisfies $d\varphi+\varphi\wedge\varphi=0$. Therefore, by Theorem 2.2, there exists a curve

$$\widetilde{\gamma}(s)=(\gamma(s);X(s),Y(s),T)\in PSH(1)$$

such that $\widetilde{\gamma}^{*}\omega=\varphi$. By the moving frame formula (3.17), we have

$$\begin{aligned} d\gamma(s)&=X(s)ds+\tau(s)Tds,\\ dX(s)&=k(s)Y(s)ds,\\ dY(s)&=-k(s)X(s)ds-Tds, \end{aligned} \tag{4.9}$$

which means

$$\begin{aligned} X(s)&=\gamma_{\xi}^{'}(s),\\ k(s)&=\langle\frac{dX(s)}{ds},Y(s)\rangle,\\ \tau(s)&=\langle\frac{d\gamma(s)}{ds},T\rangle. \end{aligned} \tag{4.10}$$

This completes the proof of the existence.

To prove the uniqueness, suppose that two horizontally regular curves γ_1 and γ_2 have the same p-curvature $k(s)$ and contact normality $\tau(s)$. The identity (4.8) shows that they must have the same Darboux derivatives

$$\widetilde{\gamma}_1^{*}\omega=\widetilde{\gamma}_2^{*}\omega.$$

Therefore, by Theorem 2.1, there exists $g\in PSH(1)$ such that $\widetilde{\gamma}_2(s)=g\circ\widetilde{\gamma}_1(s)$, hence $\gamma_2(s)=g\circ\gamma_1(s)$ for all s. This completes the proof of the uniqueness up to a group action.

4.2. **The derivation of the p-curvature and the contact normality.** In the subsection, we will compute the p-curvature and the contact normality for horizontally regular curves (Theorem 1.4) and for the geodesics in H_1 (Theorem 1.6).

Proof of Theorem 1.4. Let $\gamma(t)=(x(t),y(t),z(t))$ be a horizontally regular curve. The horizontal arc-length s is defined by

$$s(t)=\int_0^t|\gamma_{\xi}^{'}(u)|du. \tag{4.11}$$

We first observe that there is the natural decomposition

$$\begin{aligned} \gamma^{'}(t)&=(x^{'}(t),y^{'}(t),z^{'}(t))=x^{'}(t)\frac{\partial}{\partial x}+y^{'}(t)\frac{\partial}{\partial y}+z^{'}(t)\frac{\partial}{\partial z}\\ &=\underbrace{x^{'}(t)\mathring{e}_1+y^{'}(t)\mathring{e}_2}_{\gamma_{\xi}^{\prime}(t)}+\underbrace{(z^{'}(t)+xy^{'}(t)-yx^{'}(t))\frac{\partial}{\partial z}}_{\gamma_{T}^{\prime}(t)}, \end{aligned} \tag{4.12}$$

where we abuse the notations by $\frac{\partial}{\partial z} = T$. Let $\bar{\gamma}(s)$ be the reparametrization of $\gamma(t)$ by the horizontal arc-length s. Since $\gamma^{'}(t) = \bar{\gamma}^{'}(s)\frac{ds}{dt}$, by comparing with the decomposition (4.12), one has

$$\begin{aligned} \bar{\gamma}_{\xi}^{'}(s) &= \frac{dt}{ds}(x^{'}(t)\mathring{e}_1 + y^{'}(\mathring{e}_2)), \\ \bar{\gamma}_{T}^{'}(s) &= \frac{dt}{ds}\left((z^{'}(t) + xy^{'}(t) - yx^{'}(t))T\right). \end{aligned} \tag{4.13}$$

For the p-curvature, by (4.13), note that $X(s) = \frac{dt}{ds}(x^{'}(t)\mathring{e}_1 + y^{'}(t)\mathring{e}_2)$, and $Y(s) = JX(s) = \frac{dt}{ds}(x^{'}(t)\mathring{e}_2 - y^{'}(t)\mathring{e}_1)$. A straight-forward computation shows

(4.14)

$$\begin{aligned} \frac{dX(s)}{ds} &= \frac{d}{ds}\left(\frac{dt}{ds}\left(x^{'}(t), y^{'}(t), x^{'}y(t) - xy^{'}(t)\right)\right) \\ &= \left(\frac{dt}{ds}\right)^2\left(x^{''}(t), y^{''}(t), x^{''}y(t) - xy^{''}(t)\right) + \frac{d^2t}{ds^2}\left(x^{'}(t), y^{'}(t), x^{'}y(t) - xy^{'}(t)\right) \\ &= \left(x^{''}(t)\left(\frac{dt}{ds}\right)^2 + x^{'}(t)\frac{d^2t}{ds^2}\right)\mathring{e}_1 + \left(y^{''}(t)\left(\frac{dt}{ds}\right)^2 + y^{'}(t)\frac{d^2t}{ds^2}\right)\mathring{e}_2, \end{aligned}$$

so

(4.15)

$$\begin{aligned} k(s) &= \langle\frac{dX(s)}{ds}, Y(s)\rangle \\ &= -\left(x^{''}(t)\left(\frac{dt}{ds}\right)^2 + x^{'}(t)\frac{d^2t}{ds^2}\right)y^{'}(t)\frac{dt}{ds} + \left(y^{''}(t)\left(\frac{dt}{ds}\right)^2 + y^{'}(t)\frac{d^2t}{ds^2}\right)x^{'}(t)\frac{dt}{ds} \\ &= -\left(x^{''}(t)y^{'}(t) - x^{'}(t)y^{''}(t)\right)\left(\frac{dt}{ds}\right)^3 \\ &= \frac{x^{'}y^{''} - x^{''}y^{'}}{\left((x^{'})^2 + (y^{'})^2\right)^{\frac{3}{2}}}(t). \end{aligned}$$

Again by (4.13), the contact normality has to be

$$\begin{aligned} \tau(s) &= \langle\bar{\gamma}^{'}(s), T\rangle = \langle\bar{\gamma}_{T}^{'}(s), T\rangle \\ &= \frac{dt}{ds}(z^{'}(t) + xy^{'}(t) - yx^{'}(t)) \\ &= \frac{xy^{'} - x^{'}y + z^{'}}{\left((x^{'})^2 + (y^{'})^2\right)^{\frac{1}{2}}}(t), \end{aligned} \tag{4.16}$$

and the result follows. $\square$

Next we use (4.15) and (4.16) to compute the p-curvature and the contact normality for the geodesics in H_1.

Proof of Theorem 1.6. Recall [1] that the Hamiltonian system on H_1 for the geodesics is

$$\begin{aligned} \dot{x}^k(t) &= h^{kj}(x(t))\,\xi_j(t) \\ \dot{\xi}_k(t) &= -\frac{1}{2}\sum_{i,j=1}^{3}\frac{\partial h^{ij}(x)}{\partial x^k}\xi_i\xi_j,\ \ k=1,2,3, \end{aligned} \tag{4.17}$$

where

$$h^{ij}\left(x^1,x^2,x^3\right)=\begin{pmatrix} 1 & 0 & x^2 \\ 0 & 1 & -x^1 \\ x^2 & -x^1 & \left(x^1\right)^2+\left(x^2\right)^2 \end{pmatrix}.$$

So the Hamiltonian system (4.17) can be expressed by

$$\begin{aligned} &\dot{x}^1(t)=\xi_1+x^2\xi_3, \\ &\dot{x}^2(t)=\xi_2-x^1\xi_3, \\ &\dot{x}^3(t)=x^2\xi_1-x^1\xi_2+\xi_3\left[\left(x^1\right)^2+\left(x^2\right)^2\right], \\ &\dot{\xi}_1(t)=\xi_2\xi_3-x^1\xi_3^2, \\ &\dot{\xi}_2(t)=-\xi_1\xi_3-x^2\xi_3^2, \\ &\dot{\xi}_3(t)=0. \end{aligned} \tag{4.18}$$

Since $\dot{\xi}_3(t)=0$, we have $\xi_3(t)=c_3$ for some constant c_3. When $c_3=0$, one has $x(t)=(c_1t+d_1,c_2t+d_2,(c_1d_2-c_2d_1)t+d_3)$, and this implies $k(t)=0$ and $\tau(t)=0$; when $c_3>0$, one has

$$x(t)=\left(x^1(t),x^2(t),x^3(t)\right), \tag{4.19}$$

where

$$\begin{aligned} x^1(t)&=a_1\sin(2c_3t)+a_2\cos(2c_3t)+d_1, \\ x^2(t)&=-a_2\sin(2c_3t)+a_1\cos(2c_3t)+d_2, \\ x^3(t)&=(a_2d_1+a_1d_2)\sin(2c_3t)+(a_2d_2-a_1d_1)\cos(2c_3t) \\ &\quad+2c_3\left(a_1^2+a_2^2\right)t+d_3. \end{aligned}$$

Hence $k(t)=-\frac{1}{\left[\left(a_1^2+a_2^2\right)\right]^{\frac{1}{2}}}<0$ and $\tau(t)=0$; finally, when $c_3<0$, one has

$$x(t)=\left(x^1(t),x^2(t),x^3(t)\right), \tag{4.20}$$

where

$$\begin{aligned}
x^1(t) &= a_1 \sin(-2c_3 t) + a_2 \cos(-2c_3 t) + d_1 \\
x^2(t) &= a_2 \sin(-2c_3 t) - a_1 \cos(-2c_3 t) + d_2 \\
x^3(t) &= (a_1 d_1 + a_2 d_2) \sin(-2c_3 t) - (a_2 d_1 - a_1 d_2) \cos(-2c_3 t) \\
&\qquad + 2c_3 \left(a_1^2 + a_2^2\right) t + d_3.
\end{aligned}$$

Hence $k(t) = \frac{1}{\left[\left(a_1^2+a_2^2\right)\right]^{\frac{1}{2}}} > 0$ and $\tau(t) = 0$.

The calculations above show that a horizontal curve is congruent to a geodesic if it has positive constant p-curvature. Conversely, it is easy to prove that any geodesic acted by a symmetry is still a geodesic. Therefore we complete the proof of Theorem 1.6. □

Remark 4.4. Actually, the geodesics (4.19) for $c_3 > 0$ and (4.20) for $c_3 < 0$ travel along the same path with reverse direction.

5. Differential invariants of parametrized surfaces in H_1

5.1. **The proof of Theorem 1.8.** Let $F : U \to H_1$ be a normal parametrized surface with a, b, c, l and m as the coefficients in (1.9). Denote the unique lift $\widetilde{F}$ of F to $PSH(1)$ as

$$\begin{aligned}
&\widetilde{F} = \langle F(u,v); X(u,v), Y(u,v), T \rangle, \\
&X(u,v) = F_u(u,v),\ \ JX(u,v) = Y(u,v).
\end{aligned}$$

For the convenience, henceforward we simplify $F(u,v)$ by F, and $X(u,v)$ by X and so on. We first derive the Darboux derivative $\widetilde{F}^*\omega$ of $\widetilde{F}$:

By the moving frame formula (3.17),

$$\begin{aligned}
dF &= X(\widetilde{F}^*\omega^1) + Y(\widetilde{F}^*\omega^2) + T(\widetilde{F}^*\omega^3) \\
&= F_u du + F_v dv,
\end{aligned} \tag{5.1}$$

and apply on $\frac{\partial}{\partial u}$ to get

$$F_u = dF(\frac{\partial}{\partial u}) = X(\widetilde{F}^*\omega^1)(\frac{\partial}{\partial u}) + Y(\widetilde{F}^*\omega^2)(\frac{\partial}{\partial u}) + T(\widetilde{F}^*\omega^3)(\frac{\partial}{\partial u}. \tag{5.2}$$

Compare the coefficients in (5.1)(5.2) and note that $F_u = X$, we have

$$(\widetilde{F}^*\omega^1)(\frac{\partial}{\partial u}) = 1,\ \ (\widetilde{F}^*\omega^2)(\frac{\partial}{\partial u}) = (\widetilde{F}^*\omega^3)(\frac{\partial}{\partial u}) = 0. \tag{5.3}$$

Next we insert $\frac{\partial}{\partial v}$ into (5.1) and compare the coefficients, one has

$$
\begin{aligned}
(\widetilde{F}^*\omega^1)(\frac{\partial}{\partial v}) &= \langle F_v, X\rangle = a,\\
(\widetilde{F}^*\omega^2)(\frac{\partial}{\partial v}) &= \langle F_v, Y\rangle = b,\\
(\widetilde{F}^*\omega^3)(\frac{\partial}{\partial v}) &= \langle F_v, T\rangle = c.
\end{aligned}
\tag{5.4}
$$

Combine (5.3) and (5.4) to get

$$
\begin{aligned}
\widetilde{F}^*\omega^1 &= du + adv,\\
\widetilde{F}^*\omega^2 &= bdv,\\
\widetilde{F}^*\omega^3 &= cdv.
\end{aligned}
\tag{5.5}
$$

To derive $\widetilde{F}^*\omega_1^2$, we use (3.17) again and repeat the same process above. Since

$$
\begin{aligned}
dX &= Y(\widetilde{F}^*\omega_1^2) + T(\widetilde{F}^*\omega^2)\\
&= Y(\widetilde{F}^*\omega_1^2)(\frac{\partial}{\partial u})du + Y(\widetilde{F}^*\omega_1^2)(\frac{\partial}{\partial v})dv + bTdv,
\end{aligned}
\tag{5.6}
$$

and

$$
dX = dF_u = F_{uu}du + F_{uv}dv.
\tag{5.7}
$$

We obtain

$$
\begin{aligned}
(\widetilde{F}^*\omega_1^2)(\frac{\partial}{\partial u}) &= \langle F_{uu}, Y\rangle = l,\\
(\widetilde{F}^*\omega_1^2)(\frac{\partial}{\partial v}) &= \langle F_{uv}, Y\rangle = m,\\
\omega_1^2 &= ldu + mdv,\\
b &= \langle F_{uv}, T\rangle,\\
0 &= \langle F_{uv}, X\rangle = \langle F_{uu}, X\rangle = \langle F_{uu}, T\rangle.
\end{aligned}
\tag{5.8}
$$

Therefore, by (5.5) and (5.8) we have reached the Darboux derivative

$$
\widetilde{F}^*\omega = \begin{pmatrix} 0 & 0 & 0 & 0\\ du + adv & 0 & -ldu - mdv & 0\\ bdv & ldu + mdv & 0 & 0\\ cdv & bdv & -du - adv & 0 \end{pmatrix}.
\tag{5.9}
$$

By (5.9), the coefficients a, b, c, l, m uniquely determine the Darboux derivative, and it completes the proof of the uniqueness.

For the existence, suppose a, b, c and m, l are functions defined on U. Suppose ϕ is the $psh(1)$-valued 1-form defined by (5.9). Then we have

$$d\phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial a}{\partial u} & 0 & \frac{\partial l}{\partial v} - \frac{\partial m}{\partial u} & 0 \\ \frac{\partial b}{\partial u} & -\frac{\partial l}{\partial v} + \frac{\partial m}{\partial u} & 0 & 0 \\ \frac{\partial c}{\partial u} & \frac{\partial b}{\partial u} & -\frac{\partial a}{\partial u} & 0 \end{pmatrix} du \wedge dv, \tag{5.10}$$

and

$$\phi \wedge \phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -lb & 0 & 0 & 0 \\ al - m & 0 & 0 & 0 \\ -2b & -m + al & bl & 0 \end{pmatrix} du \wedge dv. \tag{5.11}$$

Thus, ϕ satisfies the integrability condition $d\phi = -\phi \wedge \phi$ if and only lf the coefficients a, b, c, l and m satisfy the integrability condition (1.10). Therefore Theorem 2.2 implies there exists a map

$$\widetilde{F}^*(u, v) = (F(u, v); X(u, v), Y(u, v), T)$$

such that $\widetilde{F}^*\omega = \phi$. Finally, the moving frame formula (3.17) implies that $F : U \to H_1$ is a map with a, b, c, l and m as the coefficients of first kind and second kind respectively.

5.2. **Invariants of surfaces.** Let $\Sigma \hookrightarrow H_1$ be a surface such that all points on Σ are non-singular. For each point $p \in \Sigma$, one can choose a normal parametrization $F : (u, v) \in U \to \Sigma$ around p such that

$$F_u = \frac{\partial F}{\partial u} = X, \tag{5.12}$$

where X is an unit vector field defining the characteristic foliation. The following lemma characterizes the normal coordinates.

LEMMA 5.1. *The normal coordinates is determined up to a transformation of the form*

$$\begin{aligned} \widetilde{u} &= \pm u + g(v) \\ \widetilde{v} &= h(v), \end{aligned} \tag{5.13}$$

for some smooth functions $g(v), h(v)$ with $\frac{\partial h}{\partial v} \neq 0$.

Proof. Suppose that $(\widetilde{u}, \widetilde{v})$ is any normal coordinates around p, i.e.,

$$F_{\widetilde{u}} = \widetilde{X}, \tag{5.14}$$

where $\widetilde{X} = \pm X$. We have the formula for the change of the coordinates

$$F_u = F_{\widetilde{u}}\frac{\partial \widetilde{u}}{\partial u} + F_{\widetilde{v}}\frac{\partial \widetilde{v}}{\partial u},\\ F_v = F_{\widetilde{u}}\frac{\partial \widetilde{u}}{\partial v} + F_{\widetilde{v}}\frac{\partial \widetilde{v}}{\partial v}. \tag{5.15}$$

Expand $F_{\widetilde{v}} = \widetilde{a}\widetilde{X} + \widetilde{b}\widetilde{Y} + \widetilde{c}\widetilde{T}$ by the orthonormal basis $\{\widetilde{X}, \widetilde{Y}, \widetilde{T}\}$. The first identity of (5.15) implies

$$\begin{aligned} X = F_u &= \widetilde{X}\frac{\partial \widetilde{u}}{\partial u} + \left(\widetilde{a}\frac{\partial \widetilde{v}}{\partial u}\widetilde{X} + \widetilde{b}\frac{\partial \widetilde{v}}{\partial u}\widetilde{Y} + \widetilde{c}\frac{\partial \widetilde{v}}{\partial u}\widetilde{T}\right)\\ &= \left(\frac{\partial \widetilde{u}}{\partial u} + \widetilde{a}\frac{\partial \widetilde{v}}{\partial u}\right)\widetilde{X} + \widetilde{b}\frac{\partial \widetilde{v}}{\partial u}\widetilde{Y} + \widetilde{c}\frac{\partial \widetilde{v}}{\partial u}\widetilde{T}. \end{aligned} \tag{5.16}$$

Since p is a non-singular point, we see that $\widetilde{c} \neq 0$ around p, so

$$\frac{\partial \widetilde{v}}{\partial u} = 0, \tag{5.17}$$

namely, $\widetilde{v} = h(v)$ for some function $h(v)$. In addition, comparing the coefficient of X, we have

$$\pm 1 = \frac{\partial \widetilde{u}}{\partial u} + \widetilde{a}\frac{\partial \widetilde{v}}{\partial u} = \frac{\partial \widetilde{u}}{\partial u}, \tag{5.18}$$

hence $\widetilde{u} = \pm u + g(v)$ for some function $g(v)$. Finally we compute

$$\det\begin{pmatrix} \frac{\partial \widetilde{u}}{\partial u} & \frac{\partial \widetilde{u}}{\partial v} \\ \frac{\partial \widetilde{v}}{\partial u} & \frac{\partial \widetilde{v}}{\partial v}\end{pmatrix} = \det\begin{pmatrix} \pm 1 & \frac{\partial g}{\partial v} \\ 0 & \frac{\partial h}{\partial v}\end{pmatrix} = \pm\frac{\partial h}{\partial v} \neq 0, \tag{5.19}$$

and the result follows. $\square$

As what did in the previous section (5.9), we can also derive the the Darboux derivative $\widetilde{F}^*\omega$ for the normal parametrization as. One obtains four 1-forms defined on Σ locally as follows:

$$\begin{aligned} &I = \widetilde{F}^*\omega^1 = du + adv, \quad II = \widetilde{F}^*\omega^2 = bdv, \quad III = \widetilde{F}^*\omega^3 = cdv\\ &IV = \widetilde{F}^*\omega_1^2 = ldu + mdv, \end{aligned} \tag{5.20}$$

where functions a, b, c, m and l are defined as (1.9). We will show that those four 1-forms are invariants under the change of coordinates.

PROPOSITION 5.2. *Suppose $\widetilde{I}, \widetilde{II}, \widetilde{III}, \widetilde{IV}$ are those defined as (5.20) with respect to the other normal coordinates $(\widetilde{u}, \widetilde{v})$. Then we have*

$$\widetilde{I} = \pm I,\ \widetilde{II} = \pm II,\ \widetilde{III} = III,\ \widetilde{IV} = IV. \tag{5.21}$$

Proof. Suppose $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{l}, \widetilde{m}$ are the coefficients of first and second kinds with respect to the normal coordinates $(\widetilde{u}, \widetilde{v})$. We point out that all such the coefficients have the same expression as in (1.9) w.r.t. the new coordinates except for $\widetilde{X} = \pm X$ and $\widetilde{Y} = J\widetilde{X} = \pm Y$.

By lemma 5.1, there exists the functions $g(v)$ and $h(v)$ such that

$$\begin{aligned}\widetilde{u} &= \pm u + g(v),\\ \widetilde{v} &= h(v).\end{aligned}$$

We compute the transformation laws of the coefficients of the fundamental forms

$$\begin{aligned}a = \langle F_v, X\rangle &= \langle F_{\widetilde{u}}\frac{\partial \widetilde{u}}{\partial v} + F_{\widetilde{v}}\frac{\partial \widetilde{v}}{\partial v}, X\rangle\\ &= \langle \pm X\frac{\partial g}{\partial v} + F_{\widetilde{v}}\frac{\partial h}{\partial v}, X\rangle\\ &= \pm\left(\frac{\partial g}{\partial v} + \frac{\partial h}{\partial v}\widetilde{a}\right).\end{aligned} \tag{5.22}$$

Similarly, we have

$$b = \pm\frac{\partial h}{\partial v}\widetilde{b},\ \ c = \frac{\partial h}{\partial v}\widetilde{c}, \tag{5.23}$$

so $F_u = \pm F_{\widetilde{u}}$ and $F_{uu} = \pm(F_{\widetilde{u}\widetilde{u}}\frac{\partial \widetilde{u}}{\partial u} + F_{\widetilde{u}\widetilde{v}}\frac{\partial \widetilde{v}}{\partial u}) = F_{\widetilde{u}\widetilde{u}}$. Thus

$$l = \pm\widetilde{l}. \tag{5.24}$$

Similarly

$$m = \frac{\partial g}{\partial v}\widetilde{l} + \frac{\partial h}{\partial v}\widetilde{m}. \tag{5.25}$$

From the transformation laws (5.22), (5.23), (5.24) and (5.25), the result (5.21) follows. □

Remark 5.3. In the previous proof (5.23), denote

$$\alpha = \frac{b}{c}, \qquad \widetilde{\alpha} = \frac{\widetilde{b}}{\widetilde{c}}, \tag{5.26}$$

then we have $\alpha = \pm\widetilde{\alpha}$. Actually, α is a function defined on the non-singular part of Σ, independent of the chooses of the normal coordinates up to a sign, such that $\alpha e_2 + T \in T\Sigma$, and hence an invariant of Σ on the non-singular part. Similarly, from (5.24), so is for l, which actually is the p-mean curvature.

Remark 5.4. We point out that the signs appearing for α and l are due to the different choices of the orientations. Indeed, if one chooses

the normal coordinates with respect to a fixed orientation of the characteristic foliation, then we will have $\alpha = \widetilde{\alpha}$ and $l = \widetilde{l}$.

Besides the invariants α and l, we now proceed the other invariant of Σ, which is globally defined on Σ, not just on the non-singular part. From Proposition 5.2, we have

$$I \otimes I + II \otimes II + III \otimes III = \widetilde{I} \otimes \widetilde{I} + \widetilde{II} \otimes \widetilde{II} + \widetilde{III} \otimes \widetilde{III}. \tag{5.27}$$

Therefore the differential form $I \otimes I + II \otimes II + III \otimes III$ again is independent of the choices of the normal coordinates, and hence also an invariant of Σ. Next we characterize the invariant.

LEMMA 5.5. *Let g_Θ be the adapted metric on H_1. Then we have*

$$g_\Theta|_\Sigma = I \otimes I + II \otimes II + III \otimes III, \tag{5.28}$$

on the non-singular part of Σ.

Proof. This lemma is a direct consequence of the moving frame formula (3.17). □

5.3. **A complete set of invariants for surfaces in H_1.** In this section, we will obtain the last invariant $IV = \widetilde{F}^*\omega_1^2$ which is completely determined by the invariants α, g_Θ, l. We therefore have a complete set of invariants for the nonsingular part of the surfaces in H_1.

Let Σ be an oriented surface and suppose $f : \Sigma \to H_1$ be an embedding. For the convenience, we will not distinguish the surfaces Σ and $f(\Sigma)$. For each non-singular point $p \in \Sigma$, we specify an orthonormal frame $(p; e_1, e_2, T)$, where e_1 is tangent to the characteristic foliation and $e_2 = Je_1$. A Darboux frame is a moving frame which is smoothly defined on Σ except for the singular points, and hence there exists a lifting of f to $PSH(1)$ defined by F. Now we would like to compute the Darboux derivative $F^*\omega$ of F. In the following, instead of $F^*\omega$, we still use

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -{\omega_1}^2 & 0 \\ \omega^2 & {\omega_1}^2 & 0 & 0 \\ \omega^3 & \omega^2 & -\omega^1 & 0 \end{pmatrix}, \tag{5.29}$$

to express the Darboux derivative. It satisfies the integrability condition $d\omega + \omega \wedge \omega = 0$, that is,

$$\begin{aligned} d\omega^1 &= \omega_1^2 \wedge \omega^2, \\ d\omega^2 &= -\omega_1^2 \wedge \omega^1, \\ d\omega^3 &= 2\ \omega^1 \wedge \omega^2, \\ d\omega_1^2 &= 0. \end{aligned} \tag{5.30}$$

Let $g_\Theta = h + \Theta^2$ be the adapted metric. From Section 5, we have $\omega^2 = \alpha\omega^3$ on the non-singular part of Σ, it is easy to see that

$$\begin{aligned} g_\Theta|_\Sigma &= \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3, \\ &= \omega^1 \otimes \omega^1 + (1+\alpha^2)\omega^3 \otimes \omega^3. \end{aligned}$$

Define

$$\begin{aligned} \hat{\omega}^1 &= \omega^1, \\ \hat{\omega}^2 &= \sqrt{1+\alpha^2}\omega^3. \end{aligned} \tag{5.31}$$

This is an orthonormal coframe of $g_\Theta|_\Sigma$ and the corresponding dual frame is

$$\begin{aligned} \hat{e}_1 &= e_1, \\ \hat{e}_2 &= e_\Sigma = \frac{\alpha e_2 + T}{\sqrt{1+\alpha^2}}. \end{aligned} \tag{5.32}$$

Let $\hat{\omega}_1^2$ be the Levi-Civita connection of $g_\Theta|_\Sigma$ with respect to the coframe $\hat{\omega}^1, \hat{\omega}^2$. By the fundamental theorem in Riemannian geometry, we have the structure equation

$$\begin{aligned} d\hat{\omega}^1 &= -\hat{\omega}_2^1 \wedge \hat{\omega}^2, \\ d\hat{\omega}^2 &= -\hat{\omega}_1^2 \wedge \hat{\omega}^1, \\ \hat{\omega}_1^2 &= -\hat{\omega}_2^1. \end{aligned} \tag{5.33}$$

The following Proposition shows that ω_1^2 is completely determined by the induced fundamental form $g_\Theta|_\Sigma$ and the functions α and l defined in (5.26).

PROPOSITION 5.6. *We have*

(5.34)
$$\begin{aligned}\omega_1^2 &= \frac{\alpha}{\sqrt{1+\alpha^2}}\hat{\omega_1^2} + \frac{l}{1+\alpha^2}\hat{\omega}^1 + \frac{e_1\alpha}{(1+\alpha^2)^{\frac{3}{2}}}\hat{\omega}^2\\ &= l\hat{\omega}^1 + \frac{2\alpha^2+(e_1\alpha)}{\sqrt{1+\alpha^2}}\hat{\omega}^2,\\ \hat{\omega}_1^2 &= \frac{\alpha}{\sqrt{1+\alpha^2}}\omega_1^2 + \frac{2\alpha}{1+\alpha^2}\hat{\omega}^2\\ &= \frac{l\alpha}{\sqrt{1+\alpha^2}}\hat{\omega}^1 + \left(2\alpha + \frac{\alpha(e_1\alpha)}{1+\alpha^2}\right)\hat{\omega}^2.\end{aligned}$$

Proof. By $\omega^2 = \alpha\omega^3$ and the second identity of (5.31), we have

$$\begin{aligned}d\omega^2 &= d\left(\frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\hat{\omega}^2\right)\\ &= d\left(\frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\right)\wedge\hat{\omega}^2 + \frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}d\hat{\omega}^2\\ &= e_1\left(\frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\right)\hat{\omega}^1\wedge\hat{\omega}^2 - \frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\hat{\omega}_1^2\wedge\hat{\omega}^1\\ &= \hat{\omega}^1\wedge\left(e_1\left(\frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\right)\hat{\omega}^2 + \frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\hat{\omega}_1^2\right),\end{aligned}$$

where we have used the second formula of the structure equation (5.33) at the third equality above. On the other hand, from the Maurer-Cartan structure equation (5.30)

$$d\omega^2 = -\omega_1^2\wedge\omega^1 = \hat{\omega}^1\wedge\omega_1^2.$$

Combine two identities above and use the Cartan lemma, we see that there exists a function D such that

$$\omega_1^2 = \frac{e_1\alpha}{(1+\alpha^2)^{\frac{3}{2}}}\hat{\omega}^2 + \frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}}\hat{\omega}_1^2 + D\hat{\omega}^1. \tag{5.35}$$

Similarly,

$$\begin{aligned}-\hat{\omega}_2^1\wedge\hat{\omega}^2 &= d\hat{\omega}^1 = d\omega^1 = \omega_1^2\wedge\omega^2\\ &= \frac{\alpha}{\sqrt{1+\alpha^2}}\omega_1^2\wedge\hat{\omega}^2.\end{aligned} \tag{5.36}$$

Again, by Cartan lemma, there exists a function A such that

$$-\hat{\omega}_2^1 = \frac{\alpha}{\sqrt{1+\alpha^2}}\omega_1^2 + A\hat{\omega}^2. \tag{5.37}$$

Finally, use (5.33) again

$$
\begin{aligned}
-\hat{\omega}_1^2 \wedge \hat{\omega}^1 &= d\hat{\omega}^2 = d\left((1+\alpha^2)^{\frac{1}{2}}\omega^3\right) \\
&= (1+\alpha^2)^{\frac{1}{2}} d\omega^3 + d(1+\alpha^2)^{\frac{1}{2}} \wedge \omega^3 \\
&= 2\alpha(1+\alpha^2)^{\frac{1}{2}}\hat{\omega}^1 \wedge \omega^3 + \frac{\alpha}{(1+\alpha^2)^{\frac{1}{2}}} d\alpha \wedge \omega^3 \\
&= \left(2\alpha + \frac{\alpha(e_1\alpha)}{1+\alpha^2}\right)\hat{\omega}^1 \wedge \hat{\omega}^2,
\end{aligned}
\tag{5.38}
$$

where we have used the third formula of (5.30) and $\hat{\omega}^2 \wedge \omega^3 = 0$. Therefore, there exists a function B such that

$$
\hat{\omega}_1^2 = \left(2\alpha + \frac{\alpha(e_1\alpha)}{1+\alpha^2}\right)\hat{\omega}^2 + B\hat{\omega}^1. \tag{5.39}
$$

By (5.35), (5.37), we get

$$
\begin{aligned}
D &= \omega_1^2(e_1) - \frac{\alpha}{\sqrt{1+\alpha^2}}\hat{\omega}_1^2(e_1) \\
&= \frac{\omega_1^2(e_1)}{1+\alpha^2} = \frac{l}{1+\alpha^2}.
\end{aligned}
$$

Similarly, by (5.35), (5.37), (5.39), we obtain

$$
\begin{aligned}
A &= \frac{2\alpha}{1+\alpha^2}, \\
B &= \frac{l\alpha}{\sqrt{1+\alpha^2}}.
\end{aligned}
\tag{5.40}
$$

These complete the proof. □

6. The derivation of the integrability condition (1.13)

Proof. We compute

(6.1)

$$
\begin{aligned}
0 &= d\omega_1^2 \\
&= d\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\hat{\omega_1^2} + \frac{l}{1+\alpha^2}\hat{\omega}^1 + \frac{e_1\alpha}{(1+\alpha^2)^{\frac{3}{2}}}\hat{\omega}^2\right) \\
&= \Big\{ -(1+\alpha^2)^{\frac{3}{2}}(e_\Sigma l) + (1+\alpha^2)(e_1e_1\alpha) - \alpha(e_1\alpha)^2 + 4\alpha(1+\alpha^2)(e_1\alpha) \\
&\qquad + \alpha(1+\alpha^2)^2 K + \alpha l(1+\alpha^2)^{\frac{1}{2}}(e_\Sigma\alpha) + \alpha(1+\alpha^2)l^2\Big\}\frac{\hat{\omega}^1 \wedge \hat{\omega}^2}{(1+\alpha^2)^{\frac{5}{2}}}.
\end{aligned}
$$

Therefore the integrability condition (1.13) is equivalent to $d\omega_1^2 = 0$. □

7. The proof of Theorem 1.10

Proof. First we show the existence. Define an $psh(1)$-valued 1-form ϕ on the non-singular part of Σ by

$$\phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \hat{\omega}^1 & 0 & -\omega_1^2 & 0 \\ \frac{\alpha'}{\sqrt{1+(\alpha')^2}}\hat{\omega}^2 & \omega_1^2 & 0 & 0 \\ \frac{1}{\sqrt{1+(\alpha')^2}}\hat{\omega}^2 & \frac{\alpha'}{\sqrt{1+(\alpha')^2}}\hat{\omega}^2 & -\hat{\omega}^1 & 0 \end{pmatrix}, \tag{7.1}$$

where

$$\omega_1^2 = \frac{\alpha'}{\sqrt{1+(\alpha')^2}}\hat{\omega_1^2} + \frac{l'}{1+(\alpha')^2}\hat{\omega}^1 + \frac{e_1\alpha'}{(1+(\alpha')^2)^{\frac{3}{2}}}\hat{\omega}^2. \tag{7.2}$$

It is easy to check that ϕ satisfies $d\phi + \phi \wedge \phi = 0$ if and only if the integrability condition (1.13) holds. Therefore, by Theorem 2.2, for each point $p \in \Sigma$ there exists an open set U containing p and an embedding $f : U \to H_1$ such that $g = f^*(g_\Theta), \alpha' = f^*\alpha$ and $l' = f^*l$. For the uniqueness, by Proposition 5.6, the Darboux derivative is completely determined by the induced metric $g_\Theta|_\Sigma$, the p-variation α and the p-mean curvature l. Therefore, by Theorem 2.1, the embedding into H_1 is unique up to a Heisenberg rigid motion. $\square$

8. Application: Crofton formula

Since the singular set in H_1 consists of isolated points and the integral over the set of isolated points has zero measure, we can always assume that there are no singular points in the context.

Definition 8.1. An **oriented horizontal line** l in H_1 is a straightly oriented line such that any point $p \in l$ the tangent vector of the line at p lies on the contact plane ξ_p. For the convenience we sometimes call a *horizontal line* or a *line*. Denote $\mathcal{L}$ by the set of all oriented horizontal lines in H_1.

Proposition 8.2. *Any horizontal line $l \in \mathcal{L}$ can be coordinatized by the triple $(p, \theta, t) \in \mathbb{R} \times S^1 \times \mathbb{R}$, and can also be parameterized by a base point $B = (p\cos\theta, p\sin\theta, t)$ with a horizontally unit-speed vector $U = (\sin\theta, -\cos\theta, p)$, namely,*

$$l(s) : (p\cos\theta, p\sin\theta, t) + s(\sin\theta, -\cos\theta, p), \forall s \in \mathbb{R}\,. \tag{8.1}$$

Proof. We consider the projection $\pi(l)$ of the line $l \in \mathcal{L}$ on the xy-plane. Since $\pi(l)$ can be uniquely determined by the pair (p, θ) where $p \in \mathbb{R}$ is the oriented distance from the origin to the line $\pi(l)$ (see [6] or

the Remark below) and $\theta \in [0, 2\pi)$ is the angle from the positive x-axis to the normal (Fig. 1.1), the points $(x, y) \in \pi(l)$ satisfy the equation

$$x \cos\theta + y \sin\theta = p. \tag{8.2}$$

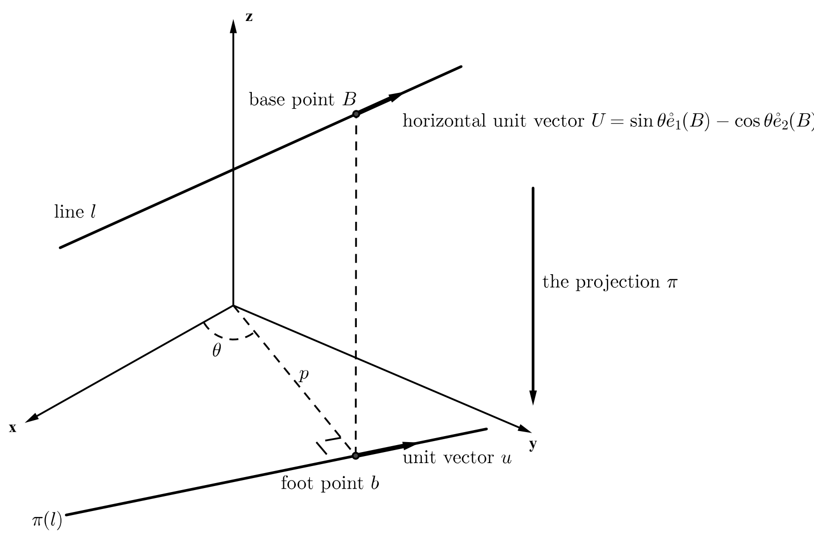

Fig. 1.1

On the projection $\pi(l)$, denote the foot point

$$b = (p\cos\theta, p\sin\theta),$$

and the unit tangent vector of $\pi(l)$ along the projection

$$u = (\sin\theta, -\cos\theta), |u|_{\mathbb{R}^2} = 1, \tag{8.3}$$

where $|u|_{\mathbb{R}^2}$ is the Euclidean length of u on the xy-plane; on the line $l \in H_1$, we have the lifting of the foot point b, called the *base point*

$$B = (p\cos\theta, p\sin\theta, t) \text{ for some } t \in \mathbb{R}\,.$$

Denote the tangent vector of l at point B by $T(B)$. Since l is horizontal, which implies that $T(B) \in \xi_B := span\{\mathring{e}_1(B), \mathring{e}_2(B)\}$, so

$$\begin{aligned} T(B) &= \alpha\mathring{e}_1(B) + \beta\mathring{e}_2(B) \\ &= \alpha(1, 0, p\sin\theta) + \beta(0, 1, -p\cos\theta) \\ &= (\alpha, \beta, \alpha p\sin\theta - \beta p\cos\theta) \end{aligned} \tag{8.4}$$

for some $\alpha, \beta \in \mathbb{R}$. Notice that the projection $\pi(T(B))$ is exactly the unit tangent vector u along the projection $\pi(l)$. Hence by comparing

the first two components of (8.4) with (8.3) we have

$$\alpha = \sin\theta,$$
$$\beta = -\cos\theta,$$

and

$$T(B) = (\sin\theta, -\cos\theta, p).$$

Therefore by defining the horizontal vector

$$U := T(B) = \sin\theta \mathring{e}_1(B) - \cos\theta \mathring{e}_2(B),$$

we have $|U|_{\xi(B)} = 1$, the horizontally unit-speed, and conclude that the line l can be uniquely determined by the triple (p, θ, t)(i.e. the base point B) and can also be parameterized by $B + sU$ for any $s \in \mathbb{R}$ as shown in (8.1). □

Remark 8.3. For our purpose of doing the integration later, the issue of the orientations are naturally involved. Actually the orientations of lines in H_1 are raised from the orientations of lines in $\mathbb{R}^2$: In [6], denote

$$\mathcal{X}_{nonoritned} := \{\text{all lines in } \mathbb{R}^2\},$$

and consider the mapping

$$\mathbb{R} \times S^1 \xrightarrow{\phi} \mathcal{X}_{nonoritned},$$

which carries (p, θ) to the line having the equation (8.2). It can be checked that $\mathcal{X}_{nonoriented}$ is a two dimensional nonoriented smooth manifold equipping the two-folds covering spaces, more precisely one has

$$\phi(p, \theta) = \phi(p', \theta')$$

if and only if either

$$p = p', \theta = \theta' \text{ or } p = -p', \theta = \theta' + \pi.$$

To our purpose, we consider the larger space

$$\mathcal{X}_{oritned} := \{(p, \theta) \in \mathbb{R} \times S^1\}$$

equipping two orientations. (Fig. 1.2)
Therefore, in H_1, instead of using the nonoriented coordinates for the set

$$\{(p, \theta, t) | p \geq 0, \theta \in [0, 2\pi), t \in \mathbb{R}\},$$

we henceforth consider the set of all horizontally oriented lines

$$\mathcal{L} := \{(p, \theta, t) \in \mathbb{R} \times S^1 \times \mathbb{R}\}$$

with two orientations.

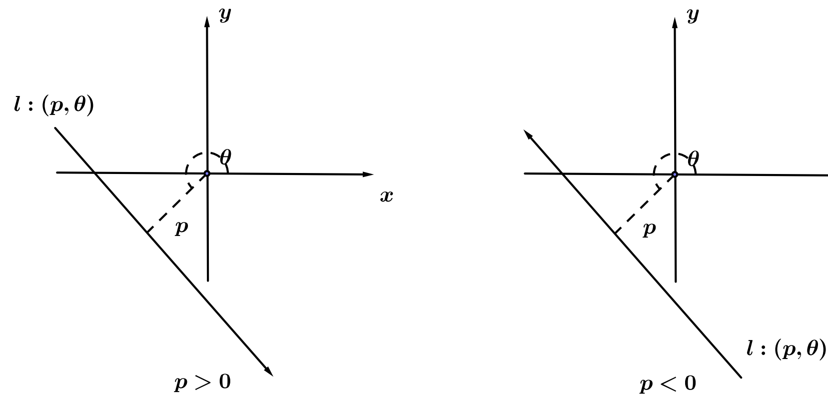

Fig. 1.2 Two orientations for the line l on $\mathbb{R}^2$.

Next, we consider the intersections of lines and a fixed regular surface $\mathbb{X} : (u,v) \in \Omega \to (x(u,v), y(u,v), z(u,v)) \in \Sigma$ embedded in H_1 for some domain $\Omega \subset \mathbb{R}^2$. To describe the position of the intersection in $\mathbb{R}^3$, one needs exact three variables. We have already known, by Proposition 8.2, a line can be represented by the triple (p, θ, t). Hence if we regard lines and surfaces as a whole system (the configuration space) and use five variables $\{(p, \theta, t, u, v)\}$ to describe the behavior of the intersections, two additional constraints are necessarily required to make the number of the freedoms be three. Those constraints can be obtained from the following Proposition.

Proposition 8.4. *Let $\mathbb{X}(u,v) = (x(u,v), y(u,v), z(u,v)) \in \Sigma$ be the parameterized regular surface in H_1. Then the configuration space D which describes the horizonal oriented lines intersecting Σ should be*

$$\begin{aligned} D = \{(p,\theta,t,u,v) \in \mathbb{R} \times S^1 \times \mathbb{R} \times \Omega \\ \mid \textit{the lines } (p,\theta,t) \in \mathcal{L} \textit{ passing through the point } \mathbb{X}(u,v) \textit{ on } \Sigma\} \\ = \{(p,\theta,t,u,v) \in \mathbb{R} \times S^1 \times \mathbb{R} \times \Omega \\ \mid \textit{the variables } p,\theta,t,u,v \textit{ satisfy } (8.5) \textit{ and } (8.6)\}, \end{aligned}$$

where

$$x(u,v)\cos\theta + y(u,v)\sin\theta = p, \tag{8.5}$$

$$z(u,v) = t + (x(u,v)\sin\theta - y(u,v)\cos\theta)p. \tag{8.6}$$

Proof. Suppose the line $l(s)$ parameterized by (8.1) intersects the surface Σ at the point q. (Fig. 1.3)

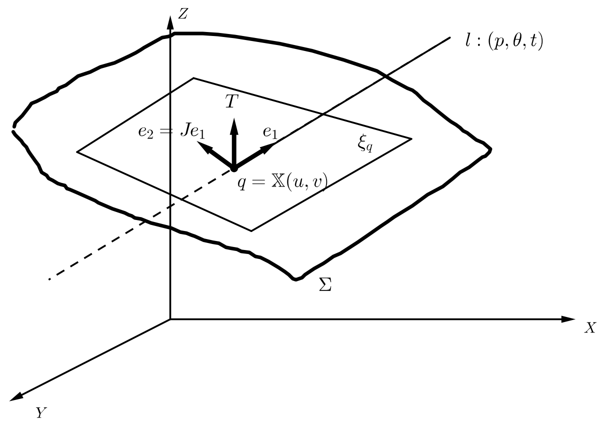

Fig. 1.3

At the point q, by Proposition 8.2, we have

$$x(u, v) = p\cos\theta + s \cdot \sin\theta, \tag{8.7}$$

$$y(u, v) = p\sin\theta - s \cdot \cos\theta, \tag{8.8}$$

$$z(u, v) = t + s \cdot p, \tag{8.9}$$

for some $s \in \mathbb{R}$. By (8.7), (8.8), one has

$$x(u, v)\cos\theta + y(u, v)\sin\theta = p,$$

which is compatible with (8.2) and we obtain the first constraint (8.5). Finally, use (8.7),(8.8) again to solve for the parameter s, and substitute s into (8.9). It is easy to have the second constraint (8.6) $\square$

Remark 8.5. By a simple calculation and (8.5), we observe that

$$\begin{aligned} U(B) &= \sin\theta\mathring{e}_1(B) - \cos\theta\mathring{e}_2(B) \\ &= \sin\theta\mathring{e}_1(\mathbb{X}(u, v)) - \cos\theta\mathring{e}_2(\mathbb{X}(u, v)) = U(\mathbb{X}(u, v)), \end{aligned}$$

i.e. the horizontally unit-speed vector field U along the line have the same vector-value when being evaluated at the based point B and at the intersection $q = \mathbb{X}(u, v)$.

Actually, the coordinates (u, v) determine where the intersections should be located on the surface, and the angle θ decides how those lines penetrate through the surface. Thus, instead of using (p, θ, t) as the coordinates for the configuration space, we can also adopt the triple $\{(u, v, \theta) \in \Omega \times S^1\}$ as the coordinates. Since the intersection q is not

only on the line but on the surface, we can derive the change of the coordinates for those coordinates.

By Remark 8.5 we choose the frame $\{\mathbb{X}(u,v); e_1(\theta), e_2(\theta), T\}$ on D where

$$\left\{\begin{array}{ll} e_1 & := \sin\theta \mathring{e}_1 - \cos\theta \mathring{e}_2, \\ e_2 & := Je_1 = \cos\theta \mathring{e}_1 + \sin\theta \mathring{e}_2, \\ T & := (0,0,1), \end{array}\right. \tag{8.10}$$

(Fig. 1.3) and denote the corresponding coframes $\{\mathbb{X}(u,v); \omega^1, \omega^2, \Theta\}$ with the connection 1-form ω_1^2.

The first formula connects the coframes and the coordinates (p, θ, t) of the line.

PROPOSITION 8.6. *If we choose the frames* $\{\mathbb{X}(u,v); e_1(\theta), e_2(\theta), T\}$, *defined by* (8.10) *and the corresponding coframes* $\{\mathbb{X}(u,v); \omega^1, \omega^2, \Theta\}$ *with the connection* 1*-form* ω_1^2, *then we have*

$$\omega^2 = dp + \langle \mathbb{X}, e_1 \rangle d\theta, \tag{8.11}$$

$$\omega_1^2 = d\theta, \tag{8.12}$$

$$\Theta = dt, \ mod\ d\theta, dp. \tag{8.13}$$

One concludes that

$$\omega^2 \wedge \omega_1^2 = dp \wedge d\theta, \tag{8.14}$$

$$\omega^2 \wedge \omega_1^2 \wedge \Theta = dp \wedge d\theta \wedge dt = \pi^* dL, \tag{8.15}$$

where π *is the projection from* D *to* $\mathcal{L}$, *and* $\langle , \rangle$ *is the Levi-metric.*

Proof. On the surface since $\mathbb{X} = (x,y,z) = x(1,0,y) + y(0,1,-x) + (0,0,z) = x\mathring{e}_1 + y\mathring{e}_2 + zT$, we have

$$\langle \mathbb{X}, e_1 \rangle = \langle x\mathring{e}_1 + y\mathring{e}_2 + zT, \sin\theta \mathring{e}_1 - \cos\theta \mathring{e}_2 \rangle = x\sin\theta - y\cos\theta.$$

Thus, by the moving frame formula (3.17) and the first constraint (8.5)

$$\begin{aligned}
\omega^2 &= \langle d\mathbb{X}, e_2\rangle = \langle dx\ \mathring{e}_1 + dy\ \mathring{e}_2 + \Theta\ \frac{\partial}{\partial z}, e_2\rangle \\
&= \cos\theta dx + \sin\theta dy \\
&= dp + (x\sin\theta - y\cos\theta)d\theta \\
&= dp + \langle \mathbb{X}, e_1\rangle d\theta; \\
\omega_1^2 &= -\omega_2^1 \\
&= -\langle de_2, e_1\rangle \\
&= \langle \sin\theta d\theta\ \mathring{e}_1 + \cos\theta d\theta\ \mathring{e}_2, \sin\theta\mathring{e}_1 + \cos\theta\mathring{e}_2\rangle \\
&= \sin^2\theta\ d\theta + \cos^2\theta\ d\theta \\
&= d\theta.
\end{aligned}$$

By the second constraint (8.6) and the parameterization of the line (8.1)

$$\begin{aligned}
\Theta &= dz + xdy - ydx \\
&= (dt + (x\sin\theta - y\cos\theta)dp + pd(x\sin\theta - y\cos\theta)) + xdy - ydx \\
&= dt + (p\sin\theta - y)dx - (p\ \cos\theta - x)dy,\ \bmod d\theta, dp \\
&= dt + s(\cos\theta dx + \sin\theta dy),\ \bmod d\theta, dp,\ \text{for some } s \in \mathbb{R} \\
&= dt\ ,\ \bmod d\theta, dp,
\end{aligned}$$

and the result follows. □

The next lemma characterize the 1-dimension foliation.

LEMMA 8.7. *Let $E = \alpha\mathbb{X}_u + \beta\mathbb{X}_v$ be the tangent vector field defined on the regular surface $\sum = \mathbb{X}(u,v)$. Then the vector E is also on the contact bundle ξ (and hence in $TH_1 \cap \xi$) if and only if pointwisely the coefficients α and β satisfy*

$$\alpha t_u + \beta t_v + x(\alpha y_u + \beta y_v) - y(\alpha x_u + \beta x_v) = 0, \tag{8.16}$$

equivalently,

$$\alpha(t_u + xy_u - yx_u) + \beta(t_v + xy_v - yx_v) = 0. \tag{8.17}$$

Proof. First, we assume that $E = \alpha\mathbb{X}_u + \beta\mathbb{X}_v = \alpha(x_u, y_u, z_u) + \beta(x_v, y_v, z_v) = c\mathring{e}_1 + d\mathring{e}_2 = (c, d, cy - dx)$ for some constants c and d. Compare each component of E to have

$$\begin{aligned}
\alpha x_u + \beta x_v &= c, \\
\alpha y_u + \beta y_v &= d, \\
\alpha z_u + \beta z_v &= cy - dx.
\end{aligned}$$

Substitute the last equation by the first two, we get the necessary condition $\alpha z_u + \beta z_v = (\alpha x_u + \beta x_v)y - (\alpha y_u + \beta y_v)x$.

The reverse part can be obtained by the direct computation

$$
\begin{aligned}
E &= (\alpha x_u + \beta x_v, \alpha y_u + \beta y_v, \alpha z_u + \beta z_v) \\
&= (\alpha x_u + \beta x_v)(1, 0, y) + (\alpha y_u + \beta y_v)(0, 1, -x) \\
&\qquad + \big(0, 0, \alpha z_u + \beta z_v - y(\alpha x_u + \beta x_v) + x(\alpha y_u + \beta y_v)\big) \\
&= (\alpha x_u + \beta x_v)\mathring{e}_1 + (\alpha y_u + \beta y_v)\mathring{e}_2.
\end{aligned}
$$

We have used the condition (8.16) in the last equality. □

The second formula for the change of coordinates connects the coframes and the coordinates of the surface.

PROPOSITION 8.8. *Suppose we choose the frames $\{\mathbb{X}(u,v); e_1(\theta), e_2(\theta), T\}$ on D and the coframes with the connection 1-form defined by above (8.10). We have the identity*

$$\Theta \wedge \omega^2 \wedge \omega_1^2 = \langle E, e_2\rangle du \wedge dv \wedge d\theta, \tag{8.18}$$

where the singular foliation

$$E := (z_u + xy_u - yx_u)\mathbb{X}_v - (z_v + xy_v - yx_v)\mathbb{X}_u \tag{8.19}$$

defines the characteristic foliation of Σ, which is induced from the contact plane ξ.

Proof. By Proposition 8.6 and the moving frame formula (3.17)

$$
\begin{aligned}
&\Theta \wedge \omega^2 \wedge \omega_1^2 \\
=&(dz + xdy - ydx) \wedge \langle d\mathbb{X}, e_2\rangle \wedge d\theta \\
=&\big((z_u + xy_u - yx_u)du + (z_v + xy_v - yx_v)dv\big) \wedge \big(\angle \mathbb{X}_u, e_2\rangle du \wedge d\theta + \angle \mathbb{X}_v, e_2\rangle dv \wedge d\theta\big) \\
=&\langle (z_u + xy_u - yx_u)\mathbb{X}_v - (z_v + xy_v - yx_v)\mathbb{X}_u, e_2\rangle du \wedge dv \wedge d\theta \\
=&\langle E, e_2\rangle du \wedge dv \wedge d\theta.
\end{aligned}
$$

To prove the vector $E \in TM \cap \xi$, it suffices to show that the coefficients $\alpha := (z_u + xy_u - yx_u)$ and $\beta := -(z_v + xy_v - yx_v)$ satisfy the condition (8.17), and we complete the proof by the previous Lemma 8.7. □

REMARK 8.9. In classical Integral Geometry [6] [16], the quantity $dL := dp \wedge d\theta \wedge dt$ is called the *(kinematic) density* of the line $(p, \theta, t) \in \mathbb{R}^3$, which is always chosen to be positive depending the orientation. Hence, according to (8.15) and (8.18), in the following proof we have to consider the orientation of $\{(u, v, \theta)\}$ to ensure the positivity of the quantity $\langle E, e_2\rangle$.

Proof of Theorem 1.11. By Remark 8.9, we choose $du \wedge dv \wedge d\theta$ as the orientation of D. Let $D = D^+ \cup D^-$, where

$$\begin{aligned} D^+ &:= \{(p, \theta, t, u, v) | \langle E, e_2 \rangle \geq 0\}, \\ D^- &:= \{(p, \theta, t, u, v) | \langle E, e_2 \rangle \leq 0\}, \\ \Gamma &:= D^+ \cap D^-. \end{aligned}$$

By the structure equation (5.30),

$$\begin{aligned} d(\Theta \wedge \omega^1) &= d\Theta \wedge \omega^1 - \Theta \wedge d\omega^1 \\ &= (2\omega^1 \wedge \omega^2) \wedge \omega^1 - \Theta \wedge (\omega_1^2 \wedge \omega^2) \\ &= \Theta \wedge \omega^2 \wedge \omega_1^2. \end{aligned} \tag{8.20}$$

We also have

(8.21)

$$\begin{aligned} \Theta \wedge \omega^1 &= (dz + xdy - ydx) \wedge \langle d\mathbb{X}, e_1 \rangle \\ &= ((z_u + xy_u - yx_u)du + (z_v + xy_v - yx_v)dv) \wedge (\langle \mathbb{X}_u, e_1 \rangle du + \langle \mathbb{X}_v, e_1 \rangle dv) \\ &= \langle E, e_1 \rangle du \wedge dv. \end{aligned}$$

Thus, integrating the kinematic density dL over the set $\mathcal{L}$, and use (8.15), (8.20), the Stock's theorem, and (8.21), which imply

(8.22)

$$\begin{aligned} \int_{l \in \mathcal{L},\ l \cap \Sigma \neq \emptyset} n(l \cap \Sigma) dL &= 2\Big(\int_{D^+} \pi^* dL - \int_{D^-} \pi^* dL \Big) \\ &= 2\Big(\int_{D^+} \omega^2 \wedge \omega_1^2 \wedge \Theta - \int_{D^-} \omega^2 \wedge \omega_1^2 \wedge \Theta \Big) \\ &= 2\Big(\int_{\partial D^+} \Theta \wedge \omega^1 - \int_{\partial D^-} \Theta \wedge \omega^1 \Big) \\ &= 2\Big(\int_{\Gamma^+ \cup \Gamma} \Theta \wedge \omega^1 - \int_{\Gamma^- \cup \Gamma} \Theta \wedge \omega^1 \Big), \end{aligned}$$

where $\Gamma^{\pm} := \partial D^{\pm} \setminus \Gamma$. We point out that for each line, there are two orientations passing through the surface, so we put double in front of the integral.

Next, we show that $du \wedge dv = 0$ on $\Gamma^{\pm}$. Indeed, by using the coordinates $\{(u, v, \theta)\}$ for the configuration space D, any vector field defined on Γ^+ can be represented by $A \wedge \frac{\partial}{\partial \theta} \in \partial\Sigma \times S^1$ for some vector A defined on the tangent bundle $T\partial\Sigma$. The value $du \wedge dv$ evaluated on Γ^+ has to be

$$du \wedge dv\Big(A \wedge \frac{\partial}{\partial \theta}\Big) = du(A) \cancel{dv\Big(\frac{\partial}{\partial \theta}\Big)}^{=0} - dv(A) \cancel{du\Big(\frac{\partial}{\partial \theta}\Big)}^{=0} = 0.$$

Therefore, (8.22) becomes

(8.23)
$$\begin{aligned}\int_{l\in\mathcal{L},\ l\cap\Sigma\neq\emptyset} n(l\cap\Sigma)dL &= 2\Big(2\int_{\Gamma}\Theta\wedge\omega^1+\int_{\Gamma^+}\Theta\wedge\omega^1-\int_{\Gamma^-}\Theta\wedge\omega^1\Big)\\ &= 4\int_{\Gamma}\Theta\wedge\omega^1\\ &= 4\int_{\Gamma}|E|du\wedge dv\\ &= 4\cdot\text{p-area}(\Sigma),\end{aligned}$$

we have used (8.21) and E is parallel to e_1 on Γ at the third equality. □

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